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ON SOME NONCENTRAL DISTRIBUTIONS IN MULTIVARIATE ANALYSIS

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In this paper, the authors derived expressions for the marginal distributions of any few consecutive ordered roots, moments of the elementary symmetric functions of the ordered roots and the Laplace transformations of the traces of a class of random matrices in the noncentral cases. This class of random matrices includes the MANOVA, Canonical Correlation, and Wishart matrices. The expressions obtained here are in terms of the linear combinations of the products of double integrals; these double integrals can be evaluated without difficulty for the cases of the random matrices that occur commonly in multivariate statistical analysis. In deriving the results presented in this paper, the authors exploited the method of de Bruijn (J. Indian Math Soc. 19 133-152) for the evaluation of certain integral.

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FOREWORD

This report was prepared for the Applied Mathematics Research Laboratory, Aerospace Research Laboratories, by P.R. Krishnaiah and A.K. Chattopadhyay under project 7071, "Research in Applied Mathematics". The work of A.K. Chattopadhyay is performed at the Aerospace Research Laboratories while in the capacity of Technology Incorporated Visiting Research Associate under contract F33615-71-C-1463, T.I. Project No. 4262B.

In this report, the authors consider the problems connected with certain non null distributions associated with the eigenvalues of a class of random matrices.

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ABSTRACT

In this paper, the authors derived expressions for the marginal distributions of any few consecutive ordered roots, moments of the elementary symmetric functions of the ordered roots and the Laplace transformations of the traces of a class of random matrices in the noncentral cases. This class of random matrices includes the MANOVA, Canonical Correlation, and Wishart matrices. The expressions obtained here are in terms of the linear combinations of the products of double integrals; these double integrals can be evaluated without difficulty for the cases of the random matrices that occur commonly in multivariate statistical analysis. In deriving the results presented in this paper, the authors exploited the method of de Bruijn (J. Indian Math Soc. **19** 133-152) for the evaluation of certain integral.

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1. INTRODUCTION

Krishnaiah and his associates [4-6] derived exact expressions for the marginal distributions of any single or few ordered roots of a class of random matrices as well as the distributions of the traces of two random matrices. They derived the above results by exploiting the method of integration over alternate variables (e.g., see [7]). For a very brief summary of the literature on the distributions of the individual roots and traces of some random matrices the reader is referred to [4-6].

In general one can use any suitable function(s)(not necessarily symmetric) of the roots of the appropriate random matrices to test various hypotheses that arise on such problems as MANOVA, canonical correlation, tests for equality of covariance matrices, etc. Of course, the choice of these functions depends upon such factors as the optimum properties of the tests and the feasibility of the evaluation of the distributions. It will be of interest to examine the distributions of some of these functions which have at least intuitive appeal in testing some of the hypotheses.

In this note, we extend the results in [4-6] to the non-central cases by using the method of de Bruijn [1] for the evaluation of certain integral. Taking advantage of de Bruijn's method [1], we have also derived the moments of the elementary symmetric functions of roots of a class of random matrices in the non-central case. Here we note that Pillai and his associates (see [2,3,9] and the references there) derived the moments of elementary symmetric functions of roots explicitly in some special cases.

2. PRELIMINARIES

de Bruijn [1] proved the following uesful result:*

Lemma 2.1 (de Bruijn). Let (a,b) be any interval finite or infinite. Then

$$\int_{a \le x_1 \le \cdots \le x_n \le b} \cdots \int \phi(x_1,\cdots,x_n)\, dx_1 \cdots dx_n = \mathrm{Pf}(A) \qquad (2.1)$$

where

$$\phi(x_1,\cdots,x_n) = |y_{ij}|,\ y_{ij} = \phi_i(x_j),$$

$A = (a_{ij})$ and Pf(A) denotes the Pfaffian of A. Here the elements a_{ij} are given by

$$a_{ij} = \int_a^b \int_a^b \phi_i(x)\, \phi_j(y)\, \mathrm{Sgn}(y - x)\, dx\, dy \qquad i,j = 1,\cdots,2m \qquad (2.2)$$

if $n = 2m$; if $n = 2m + 1$, then they are given by (2.2) and by

$$a_{2m+2,\ 2m+2} = 0,$$

$$a_{i,\ 2m+2} = -a_{2m+2,\ i} = \int_a^b \phi_i(x)\, dx \qquad i = 1,\cdots,2m+1 .$$

We need the following in the sequel:

Lemma 2.2. Let $\eta(x_1,\cdots,x_n)$ be a symmetric function of $x_1,\cdots, x_n$ and $\phi(x_1,\cdots,x_n)$ be as defined before. Then

* The integral in Eq. (2.2) of [6] is a special case of the integral in Eq. (2.1) of this paper.

$$\int_{a \le x_1 \le \cdots \le x_n \le b} \cdots \int \eta(x_1,\cdots,x_n)\ \phi(x_1,\cdots,x_n)\ dx_1 \cdots dx_n$$

$$= \int_a^b \cdots \int_a^b E(x_1,\cdots,x_n)\ \eta(x_1,\cdots,x_n) \prod_{i=1}^{n} \{\phi_i(x_i)\ dx_i\} \tag{2.3}$$

where

$$E(x_1,\cdots,x_n) = \prod_{1 \le j < i \le n} \mathrm{Sgn}(x_i - x_j) .$$

In addition, if

$$\eta(x_1,\cdots,x_n) = \textstyle\sum_1 C_\kappa\ x_1^{k_1} \cdots x_n^{k_n} , \tag{2.3a}$$

where C_κ depends upon $k_1,\cdots,k_n$ and $\sum_1$ denotes the summation over the different elements of $\kappa = (k_1,\cdots,k_n)$ subject to suitable restrictions, we have

$$\int_{a \le x_1 \le \cdots \le x_n \le b} \cdots \int \eta(x_1,\cdots,x_n)\ \phi(x_1,\cdots,x_n)\ dx_1 \cdots dx_n$$

$$= \textstyle\sum_1 C_\kappa\ \mathrm{Pf}(A\kappa) \tag{2.4}$$

where $A\kappa = (a_{ij}^{\kappa})$.

In the above lemma, a_{ij}^{κ} are given by

$$a_{ij}^{\kappa} = \int_a^b \int_a^b x^{k_i} \phi_i(x)\, y^{k_j} \phi_j(y) \quad \mathrm{Sgn}(y - x)\, dx\, dy \qquad (2.5)$$

$$i,j = 1,\cdots,2m$$

when $n = 2m$; if $n = 2m + 1$, the elements a_{ij}^{κ} are given by (2.5) and

$$a^{\kappa}_{2m+2,\ 2m+2} = 0$$

$$a^{\kappa}_{i,\ 2m+2} = -a^{\kappa}_{i,\ 2m+2} = \int_a^b x^{k_i} \phi_i(x)\, dx \qquad i = 1,\cdots,2m + 1.$$

Lemma 2.2 can be proved by following the same lines as in de Bruijn [1].

3. MOMENTS OF ELEMENTARY SYMMETRIC FUNCTIONS OF ROOTS

Let $\ell_1,\cdots,\ell_p$ be the latent roots of a random matrix and let us consider the situations when their joint density in the non-central case is of the form

$$f(\ell_1,\cdots,\ell_p) = C \prod_{i=1}^{p} \psi(\ell_i) \prod_{i>j} (\ell_i - \ell_j) \ \sum_2 \ \sum_1 a(\kappa) \ \eta_\kappa(L) \qquad (3.1)$$

$$a \leq \ell_1 \leq \cdots \leq \ell_p \leq b$$

where $L = \mathrm{diag}\,(\ell_1,\cdots,\ell_p)$, $\kappa = (k_1,\cdots,k_p)$ is a partition of k subject to suitable restrictions, $\eta_\kappa(L)$ is a symmetric function and $a(\kappa)$ depends upon the population parameters and the elements of the partition κ. Also $\sum_1$ denotes the summation over all partitions of k, whereas $\sum_2$ denotes the summation over k. Now, let

$$\eta_\kappa(L) = \textstyle\sum_3 b_{\underset{\sim}{r}}^\kappa \ell_1^{r_1}\cdots\ell_p^{r_p} \tag{3.2}$$

where $\sum_3$ denotes the summation over $r_1,\cdots,r_p$ subject to some suitable restrictions and b_r^κ depends on $r_1,\cdots,r_p$ and κ. If $\eta_\kappa(L)$ is the zonal polynomial and the elements of the partition $\kappa = (k_1,\cdots,k_p)$ are subject to the restriction $k_1 \geq \cdots \geq k_p \geq 0$, then $\eta_\kappa(L)$ is of the form (3.2).

The elementary symmetric function of order q is given by

$$\begin{aligned}\zeta(\ell_1,\cdots,\ell_p) &= \sum_{i_1<\cdots<i_q} \ell_{i_1}\cdots\ell_{i_q} \\ &= \textstyle\sum_4 \ell_{i_1}\cdots\ell_{i_q} \qquad \text{say}\end{aligned}$$

Now

$$\{\zeta(\ell_1,\cdots,\ell_p)\}^s = \textstyle\sum_5 \binom{s}{s_1,\cdots,s_{p^*}} \ell_1^{n_1}\cdots\ell_p^{n_p}, \tag{3.3}$$

where

$$p^* = \binom{p}{q} \text{ and } \textstyle\sum_5$$

denotes the summation over $s_1,\cdots,s_{p^*}$ such that $s_1 + \cdots + s_{p^*} = s, n_1 + \cdots + n_p = qs$, and each n_i depends upon $s_i,\cdots,s_{p^*}$. We know that

$$\prod_{i=1}^p \psi(\ell_i)\, \eta_\kappa(L)\, \{\zeta(\ell_1,\cdots,\ell_p)\}^s$$

is a symmetric function of $\ell_1,\cdots,\ell_p$. So by (2.3) and (2.4) we get the sth moment of $\zeta(\ell_1,\cdots,\ell_p)$ as

$$\mathcal{E}[\{\zeta(\ell_1,\cdots,\ell_p)\}^s] = C \sum\nolimits_2 \sum\nolimits_1 a(\kappa) \int_a^b \cdots \int_a^b E(\ell_1,\cdots,\ell_p)$$

$$\prod_{i=1}^{p} \{\psi(\ell_i)\, \ell_i^{i-1}\}\, \eta_\kappa(L)\, (\zeta(\ell_1,\cdots,\ell_p))^s \; d\ell_1\cdots d\ell_p$$

$$= C \sum\nolimits_2 \sum\nolimits_1 \sum\nolimits_3 \sum\nolimits_5 a(\kappa)\, b^{\kappa}_{\underset{\sim}{r}}\binom{s}{s_1,\cdots,s_{p*}}$$

$$(A(r_1,\cdots,r_p;\, n_1,\cdots,n_p)) \tag{3.4}$$

where

$$A(r_1,\cdots,r_p;\, n_1,\cdots,n_p) = \int_a^b \cdots \int_a^b E(x_1,\cdots,x_p)$$

$$\prod_{i=1}^{p} \{\psi(x_i)\, x^{i+r_i+n_i-1}\, dx_i\}$$

$$= \mathrm{Pf}(a_{ij}(r_1,\cdots,r_p;\, n_1,\cdots,n_p))$$

and

$$a_{ij}(r_1,\cdots,r_p;\, n_1,\cdots,n_p) = \int_a^b \cdots \int_a^b x^{i+r_i+n_i-1}\, y^{j+r_j+n_j-1}$$

$$\psi(x)\, \psi(y)\, \mathrm{sgn}(y-x)\, dx\, dy$$

when $p = 2m$; when $p = 2m+1$, the elements $a_{ij}(r_1,\cdots,r_p;\, n_1,\cdots,n_p)$ are as given above and

$$a_{2m+2,\ 2m+2}(r_1,\cdots,r_p;\ n_1,\cdots,n_p) = 0$$

$$a_{i,\ 2m+2}(r_1,\cdots,r_p;\ n_1,\cdots,n_p)$$

$$= -a_{2m+2,\ i}(r_1,\cdots,r_p;\ n_1,\cdots,n_p)$$

$$= \int_a^b x^{r_i+n_i+i-1}\ \psi(x)\ dx, \qquad i = 1,\cdots,2m+1.$$

The summations $\sum_1$, $\sum_2$, $\sum_3$, $\sum_5$, are defined in (3.1), (3.2) and (3.3).

In the central case, let us assume that the joint density of $\ell_1,\cdots,\ell_p$ is of the form

$$f(\ell_1,\cdots,\ell_p) = C \prod_{i=1}^{p} \psi(\ell_i) \prod_{i>j} (\ell_i - \ell_j).$$

In this case

$$\mathcal{E}\{t(\ell_1,\cdots,\ell_p)\}^s = C \sum_5 \binom{s}{s_1,\cdots,s_{p*}} Pf(A(0,\cdots,0;\ n_1,\cdots,n_p))$$

where $A(0,\cdots,0;n_1,\cdots,n_p)$ is given by (3.4).

4. DISTRIBUTION OF A FEW ORDERED ROOTS

We need the following in the sequel.

Remark 4.1. Let (a,b) be any interval finite or infinite. Also, let

$$I(k_1,\cdots,k_n;\ a,b) = \int \cdots_R \int \phi\ (x_1,\cdots,x_n;\ k_1,\cdots,k_n)\ dx_1\cdots dx_n$$

where

$$R: \quad a \le x_1 \le \cdots \le x_n \le b$$

and

$$\phi(x_1,\cdots,x_n;\ k_1,\cdots,k_n) = \begin{vmatrix} \phi_{k_1}(x_1) & \cdots & \phi_{k_1}(x_n) \\ \vdots & & \vdots \\ \phi_{k_1}(x_1) & \cdots & \phi_{k_1}(x_n) \end{vmatrix}$$

Then

$$I(k_1,\cdots,k_n;\ a,b) = |\ D(k_1,\cdots,k_n)\ |^{\frac{1}{2}} \tag{4.1}$$

where

$$D(k_1,\cdots,k_n) = (d_{ij}(k_1,\cdots,k_n)),$$

$$d_{ij}(k_1,\cdots,k_n) = \int_a^b \int_a^b \phi_{k_i}(x)\ \phi_{k_j}(y)\ \mathrm{Sgn}(y-x)\ dx\ dy \tag{4.2}$$

$$i,j = 1,\cdots,2m$$

when $n = 2m$; if $n = 2m+1$, we have, in addition to (4.2),

$$d_{2m+2,\ 2m+2}(k_1,\cdots,k_n) = 0$$

$$d_{i,2m+2}(k_1,\cdots,k_n) = -d_{2m+2,\ i}(k_1,\cdots,k_n) = \int_a^b \phi_{k_i}(x)\ dx$$

$$i = 1,\cdots,2m+1\ .$$

Formula (4.1) is implicit in lemma 2.1.

Let (a,b) be any interval finite or infinite and let

$$R^*:\quad a \le x_1 \le \cdots \le x_r \le x_{r+1} \le x_{r+s} \le x_{r+s+1} \le \cdots \le x_p \le b$$

Then

$$\int \cdots \int_{R^*} \phi(x_1,\cdots,x_p)\, \eta(x_1,\cdots,x_p)\, dx_1\cdots dx_r\, dx_{r+s+1}\cdots dx_p$$

$$= \sum\nolimits_6 \sum\nolimits_7 (-1)^{\frac{r(r+1)}{2} + \frac{s(s+1)}{2} + \sum_{i=1}^{r} k_i + \sum_{i=1}^{s} \alpha_i}$$

$$V(x_{r+1},\cdots,x_{r+s};\, \alpha_1,\cdots,\alpha_s) \int \cdots \int_{R^{**}} E(x_1,\cdots,x_r)\, E(x_{r+s+1},\cdots,x_p) \quad (4.3)$$

$$V(x_1,\cdots,x_r;\, k_1,\cdots,k_r)\, V(x_{r+s+1},\cdots,x_p;\, \beta_{r+s+1},\cdots,\beta_p)$$

$$\eta(x_1,\cdots,x_p)\, dx_1\cdots dx_r\, dx_{r+s+1}\cdots dx_p;\ a \le x_{r+1} \le \cdots \le x_{r+s} \le b,$$

and $(k_1 < \cdots < k_r)$ is a subset of $(1,\cdots,p)$, $(t_1 < \cdots < t_{p-r})$ is its complementary set. Also $(\alpha_1 < \cdots < \alpha_s)$ is a subset of $(t_1 < \cdots < t_{p-r})$ and $(\beta_{r+s+1} < \cdots < \beta_p)$ is the complementary subset of $(\alpha_1 < \cdots < \alpha_s)$ with respect to $(t_1 < \cdots < t_{p-r})$, $\sum_6$ denotes this summation over all possible $\binom{p}{r}$ choices of $(k_1,\cdots,k_r)$ and $\sum_7$ stand for the summation over $\binom{p-r}{s}$ possible choice of $(\alpha_1 < \cdots < \alpha_s)$,

$$V(x_1,\cdots,x_r;\, k_1,\cdots,k_r) = \det(\phi_{k_i}(x_j))_{i,j=1,\cdots,r}$$

and

$$R^{**}:\quad \{a \le x_i \le x_{r+1},\ (i=1,\cdots,r);\ x_{r+s} \le x_j \le b,\ (j = r+s+1,\cdots,p)\}.$$

Formula (4.3) follows by repeated use of Laplace expansion of $\phi(x_1,\cdots,x_p)$ starting with the first r columns and then with the first s columns of the remaining determinant. Now again if $\eta(x_1,\cdots,x_p)$ is of the form $\eta(x_1,\cdots,x_p) = \sum_8 C_\delta\, x_1^{d_1}\cdots x_p^{d_p}$ such that $\eta(x_1,\cdots,x_p)$ is symmetric in $x_1,\cdots,x_p$ and $\sum_8$ is similarly defined as in (2.3a)

$$\int\cdots\int_{R^*} \phi(x_1,\cdots,x_p)\, \eta(x_1,\cdots,x_p)\, dx_1\cdots dx_r\, dx_{r+s+1}\cdots dx_p$$

$$= \sum_8 \sum_6 \sum_7 (-1)^{\frac{r(r+1)}{2} + \frac{s(s+1)}{2} + \sum_{i=1}^{r} k_i + \sum_{i=1}^{s} \alpha_i} C_\delta$$

$$V(x_{r+1},\cdots,x_{r+s};\alpha_1,\cdots,\alpha_s)\, x_{r+1}^{d_{r+1}}\cdots x_{r+s}^{d_{r+s}}$$

$$I(k_1,\cdots,k_r,d_1,\cdots,d_r;\ a,x_{r+1})$$

$$I(\beta_{r+s+1},\cdots,\beta_p,d_{r+s+1},\cdots,d_p;\ x_{r+s},b) \qquad (4.4)$$

Where $\sum_8$ is given in (2.3a) and $I(k_1,\cdots,k_r,d_1,\cdots,d_r;\ a,b)$ is obtained from $I(k_1,\cdots,k_r;\ a,b)$ replacing $\phi_{k_i}(x_j)$ with $x_j^{d_j}\ \phi_{k_i}(x_j)$.

Now let us take the joint density of the roots in the noncentral case as given by (3.1). Then applying formula (4.3) we get the density of roots $x_{r+1},\cdots,x_{r+s}$ as

$$f(x_{r+1},\cdots,x_{r+s})$$

$$= C \sum_2 \sum_1 \sum_3 \sum_6 \sum_7 (-1)^{\frac{r(r+1)}{2} + \frac{s(s+1)}{2} + \sum_{i=1}^{r} k_i + \sum_{i=1}^{s} \alpha_i}$$

$$a(\kappa)\, b_{\underline{\lambda}}^{\kappa}\, I(k_1,\cdots,k_r,\lambda_1,\cdots,\lambda_r;\, a, x_{r+1})$$

$$I(\beta_{r+s+1},\cdots,\beta_p,\lambda_{r+s+1},\cdots,\lambda_p;\, x_{r+s}, b)$$

$$V(x_{r+1},\cdots,x_{r+s};\, \alpha_1,\cdots,\alpha_s)$$

$$x_{r+1}^{d_{r+1}} \cdots x_{r+s}^{d_{r+s}}, \qquad a \le x_{r+1} \le \cdots \le x_{r+s} \le b$$

where

$$\phi(x_1,\cdots,x_p) = \det\, (x_j^{i-1})_{i,j=1,\cdots,p} \tag{4.5}$$

$C, \sum_1, \sum_2, \sum_3, \sum_6, \sum_7$ are all defined earlier b_{λ}^{κ} and $\lambda_1,\cdots,\lambda_p$ are defined in the same way as $b_{\underline{r}}$, $r_1,\cdots,r_p$ in (3.2).

5. PROBABILITY INTEGRALS OF ORDERED ROOTS

As before, let us assume that the joint density of the roots $\ell_1 < \cdots < \ell_p$ is given by (3.1). Then the density functions of the extreme roots can be obtained by following the same lines as in Krishnaiah and Chang [5] and using (2.3) and (2.4). Also the c.d.f. of the individual extreme root can be computed by applying (2.3) and (2.4). The probability integral associated with the joint density of any two ordered roots is easily obtained by following the same lines as in Krishnaiah and Waikar [6] and applying formulae (2.3) and (2.4). The expressions associated with the density functions

and c.d.f.s of the extreme roots and the probability integral associated with any two ordered roots are not given here for the sake of brevity. We will now derive the c.d.f.s associated with the intermediate roots.

It is known that

$$P[\ell_r \leq x] = P[\ell_{r+1} \leq x] + P[\ell_1 < \cdots < \ell_r < x < \ell_{r+1} < \ell_p] \quad (5.1)$$

Now c.d.f.s of the extreme roots can be evaluated using (2.3) and (2.4). The second term in (5.1) can be calculated following similar lines as in Krishnaiah and Waikar [6] and using (4.3) and (4.4). It is given by

$$P[\ell_1 < \cdots < \ell_r < x < \ell_{r+1} < \cdots < \ell_p]$$

$$C \textstyle\sum_2 \sum_1 \sum_3 \sum_6 (-1)^{\frac{1}{2}r(r+1) + \sum_{i=1}^{r} k_i} b_{\underset{\sim}{\lambda}}^{\kappa} a(\kappa)$$

$$I(k_1,\cdots,k_r,\lambda_1,\cdots,\lambda_r;\ a,x)\ I(t_1,\cdots,t_{p-r},t_{r+1},\cdots,\lambda_p;\ x,b)$$

where

$$C,\ \textstyle\sum_1,\ \sum_2,\ \sum_3,\ \sum_6,\ (k_1 < \cdots < k_r),\ (t_1 < \cdots < t_{p-r})$$

and

$$\phi(\ell_1,\cdots,\ell_r)$$

are as defined in (4.5).

6. NON NULL DISTRIBUTIONS OF TRACES

Here we use the results in [4] and treat both traces discussed there in a unified way. Let the joint density of the roots be given as (3.1). Then

$$E(e^{-t(\ell_1 + \cdots + \ell_p)})$$

$$= C \sum_2 \sum_1 \sum_3 a(\kappa)\, b_{\underset{\sim}{r}}^{k}\, Pf(B_{\underset{\sim}{r}})$$

where

$$B_{\underset{\sim}{r}} = (b_{\underset{\sim}{r},ij})_{i,j=1,\cdots,p}$$

and

$$b_{\underset{\sim}{r},ij} = \int_a^b \int_a^b x^{r_i+i-1} y^{r_j+j-1}\, \psi(x)\, \psi(y)\, e^{-tx}\, e^{-ty}\, \mathrm{sgn}(y - x)\, dx\, dy \qquad i,j=1,\cdots,p$$

If p is odd, the $B_{\underset{\sim}{r}}$ matrix has to be augmented as given in Section 2. Now to find the exact distribution we have to invert this Laplace transform. To this end we note that

$$Pf(B_{\underset{\sim}{r}}) = \sum \pm\, b_{\underset{\sim}{r},i_1,i_2} \cdots\cdots b_{\underset{\sim}{r},i_{2a-1},i_{2a}}$$

Where the summation is over all possible choices of $i_1,\cdots,i_{2a}$ subject to this restriction $i_1 < i_2,\cdots, i_{2a-1} < i_{2a}$ and the sign is +ve or -ve according as the permutation is even or odd. Thus inverting the Laplace transform we get

$$f(V) = C \sum_2 \sum_1 \sum_3 \sum_6 a(\kappa)\, b_{\underline{r}}\, h_{\underline{r}}\, (i_1, \cdots, i_{2a};\, V)$$

where $h_{\underline{r}}(i_1, \cdots, i_{2a})$ is the inverse Laplace transform of $Pf(B_{\underline{r}})$. For discussion on inverse transform of the above type for two important cases we refer to [4].

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